Paradigms for non-classical substitutions

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Abstract

We will present three paradigms for non-classical substitution. Firstly, we have the classical substitution of variables with terms. This is written in a strict categorical form supporting presentation of the other two paradigms. The second paradigm is substitutions of variables with many-valued sets of terms. These two paradigms are based on functors and monads over the category of sets. The third paradigm is the substitution of many-valued sets of variables with terms over many-valued sets. The latter is based on functors and monads over the category of many-valued sets. This provides a transparency of the underlying categories and also makes a clear distinction between set-theoretic operations in the meta language and operations on sets and many-valued sets as found within respective underlying categories.

1. Categorical prerequisite

Let Set be the category of sets and mappings, and Set(L), where L is a completely distributive lattice, be the category with objects being pairs (A, α) where α: A → L and morphisms (A, α) → (B, β) are mappings f: A → B such that β(f(a)) ≥ α(a) for all a ∈ A. Note that Set is not isomorphic to Set(2), where 2 = {0, 1}. The category Set(L) is topological over Set and thus has all limits and colimits. It is cartesian closed. If L has in addition a semigroup operation which distributes over ∨ then Set(L) has a monoidal closed structure.

A monad (or triple, or algebraic theory) over a category C is written as Φ = (Φ, η, µ), where Φ: C → C is a (co-variant) functor, and η: id → Φ and µ: Φ ◦ Φ → Φ are natural transformations for which µ ◦ Φµ = µ ◦ µΦ and µ ◦ Φη = µ ◦ ηΦ = idΦ hold.

A Kleisli category CΦ for a monad Φ over a category C is given with objects in CΦ being the same as in C, and morphisms being defined as HomCΦ(X, Y) = HomC(X, ΦY). Morphisms f: X → Y in CΦ are thus morphisms f: X → ΦY in C, with ηX: X → ΦX being the identity morphism. Composition of morphisms in CΦ is defined as

\[(X \xrightarrow{f} Y) \circ (Y \xrightarrow{g} Z) = X \xrightarrow{f \circ g \Phi} ΦZ.\]

Morphisms in CΦ can be seen as general variable substitutions.
2. Variables substituted by terms

For a set $A$, the constant set functor $A_{\text{set}}$ is the covariant set functor which assigns sets $X$ to $A$, and mappings $f$ to the identity map $id_A$. The coproduct $\coprod_{i \in I} \varphi_i$ of covariant set functors $\varphi_i$ assigns to each set $X$ the disjoint union $\bigcup_{i \in I} \{i \times \varphi_i X\}$, and to each morphism $f : X \to Y$ in the mapping $(i, m) \mapsto (i, \varphi_i f(m))$, where $(i, m) \in (\coprod_{i \in I} \varphi_i) X$.

Let $k$ be a cardinal number and $(\Omega_n)_{n \leq k}$ be a family of sets. We will write $\Omega_n id^n$ instead of $(\Omega_n)_{\text{set}} \times id^n$. Note that $\prod_{n \leq k} \Omega_n id^n X$ is the set of all triples $(n, \omega, (x_i)_{i \leq n})$ with $n \leq k, \omega \in \Omega_n$ and $(x_i)_{i \leq n} \in X^n$.

A disjoint union $\Omega = \bigcup_{n \leq k} \{n\} \times \Omega_n$ is an operator domain, and an $\Omega$-algebra is a pair $(X, (s_n)_n)_{(n, \omega) \in \Omega}$ where $s_n : X^n \to X$ are $n$-ary operations. The $\prod_{n \leq k} \Omega_n id^n -$morphisms between $\Omega$-algebras are precisely the homomorphisms between the algebras.

The term functor can now be defined by transfinite induction. In fact, let $T_0$ be the least cardinal greater than $k$ and $\Omega_0$. Clearly, $(n, \omega, (x_i)_{i \leq n}) \in T_0 X$, $i \neq 0$, implies $m_i \in T_0 X$, $\kappa_i \leq \omega$.

$T_0$ can be extended to a monad $T = (T, \eta_T, \mu_T)$ [8].

Paradigm 1 Morphisms $f : X \to Y$ in $\text{Set}_{L_{id}}$ capture the notion of variables being substituted by terms.

3. Variables substituted by many-valued sets of terms

The many-valued extension $L_{id}$ of the powerset monad is obtained by $L_{id} X = L X$, i.e. the set of mappings $A : X \to L$. The partial order $\leq$ on $L_{id} X$ is given pointwise. Morphism $f : X \to Y$ in $\text{Set}$ are mapped according to $L_{id} f(A)(y) = \bigvee_{f(x) = y} A(x)$.

Further $\eta_X : X \to L_{id} X$ is given by $\eta_X(x)(x')$ being 1, if $x' \leq x$, and 0 otherwise, $\mu_X : L_{id} X \circ L_{id} X \to L_{id} X$ is given by $\mu_X(\mathcal{M})(x) = \bigvee_{x \in L_{id} X} A(x) \land \mathcal{M}(A)$. $L_{id} = (L_{id}, \leq, \eta, \mu)$ is a monad [8].

The composition $L_{id} \circ T_0$ can be extended to a monad. We briefly outline the constructions presented in [3]. We will need the ‘swapping’ $\sigma_X : T_0 L_{id} X \to L_{id} T_0 X$ where $\sigma_X |_{T X} = id_{L X}$, and further, for $l = (n, \omega, (x_i)_{i \leq n}) \in T^n L X$, $\alpha > 0$, $l_i \in T^\beta L X$, $\beta < \alpha$, let

$$\sigma_X(l)((n', \omega', (m_i)_{i \leq n'})) = \begin{cases} \bigwedge_{i \leq n} \sigma_X(l_i)(m_i) & \text{if } n = n' \text{ and } \omega = \omega' \\ 0 & \text{otherwise} \end{cases}$$

The partial order $\leq$ on $L_{id} X \circ T_0 X$ is constructed as follows

$$\eta_{L T} : id \to L T \quad \text{and} \quad \mu_{L T} : LTL T \to L T$$

are defined as follows

$$\eta_{L T} = \eta T \circ \eta T \quad \mu_X^{L T} = L \mu_X T \circ \mu^{L T} T \circ L \sigma_{T X}$$

Proposition 1 [3] $(L_{id} \circ T_0, \eta_{L_{id} \circ T_0}, \mu^{L_{id} \circ T_0})$, denoted $L_{id} \bullet T_0$, is a monad.

Paradigm 2 Morphisms $f : X \to Y$ in $\text{Set}_{L_{id} \bullet T_0}$ capture the notion of variables being substituted by many-valued sets of terms.

4. Many-valued set of variables substituted by terms over a many-valued set of variables

The term monad over $\text{Set}(L)$ is constructed as follows. Let $id^n(A, \alpha) = (\{\emptyset\}, \top)$ and $id^n(A, \alpha) = (id^0 A, id^n(\alpha))$, where $id^n(\alpha)(a_1, \ldots, a_n) = \bigwedge_{i=1}^n \alpha(a_i)$. Again a constant functor $(A, id(A, \alpha))_{\text{set}(L)}$ assigns any $(X, \xi)$ to $(A, \alpha)$ and all morphisms $f : (X, \xi) \to (Y, \eta)$ to the identity morphism $id(A, \alpha)$. If $(A_i, \alpha_i) \mid i \in I$ is a family of objects in $\text{Set}(L)$ then the coproduct is $\bigcup_{i \in I} A_i, \alpha_i$.

Let $k$ be a cardinal number and $(\Omega_n, \theta_n) \mid n \leq k$ be a family of $L$-sets. We have

$$\prod_{n \leq k} (\Omega_n, \theta_n)_{\text{set}(L)} \times id^n X, \beta)$$

where $\beta(n, \omega, (x_i)_{i \leq n}) = \theta_n(\omega) \land id^n(\xi)((x_i)_{i \leq n})$, $\omega \in \Omega_n$ and $(x_i)_{i \leq n} \in X^n$.

Consider $(\Omega, \theta) = \bigcup_{n \leq k} (\Omega_n, \theta_n)$ as a fuzzy operator domain, i.e. $\theta_n : \Omega_n \to L$. Again we use transfinite induction, start with $T_0 = id, T_1 = \top$ be the right side of the equation (1) and then define

$$T_{(\Omega, \theta)}(X, \xi) = \bigvee_{i \leq k} T_{(\Omega, \theta)}(X, \xi)$$

for each ordinal $i > 1$. Finally, let $T_{(\Omega, \theta)}(X, \xi) = \bigvee_{i \leq k} T_{(\Omega, \theta)}(X, \xi)$, where $k$ is the least cardinal greater
than $k$ and $\varkappa_0$. Notice that $T^{(\Omega, 0)}_{(\Omega, 0)}, T^{(1, 0)}_{(1, 0)} : \text{Set}(L) \to \text{Set}(L)$ and $\bigvee_{\iota < \kappa} T^{(\Omega, 0)}_{(\Omega, 0)}(X, \xi)$ denotes the co-limit for the family $\{T^{(\Omega, 0)}_{(\Omega, 0)}(X, \xi) \mid \iota < \kappa\}$.

For each positive ordinal $\iota$ there exists $\alpha_{\iota}$, such that $T^{(1, 0)}_{(1, 0)}(X, \xi) = (T^{(1, 0)}_{(1, 0)}X, \alpha_{\iota})$, and there exists $\alpha$ such that $T^{(1, 0)}_{(1, 0)}(X, \xi) = (T^{(1, 0)}_{(1, 0)}X, \alpha)$.

**Proposition 2** [4] $T^{(1, 0)}_{(1, 0)}$ is a functor and extendable to a monad $T^{(1, 0)}_{(1, 0)}$.

**Paradigm 3** Morphisms $f : (X, \xi) \to (Y, \chi)$ in $\text{Set}(L)_{T^{(1, 0)}_{(1, 0)}}$ capture the notion of a many-valued set of variables being substituted by terms over a many-valued set of variables.

The functors $L_{id}$ and $T_{\Omega}$ take $\text{Set}$ to $\text{Set}$. There is a lifting of $L_{id}$ to a covariant powerobject functor $\mathcal{L} : \text{Set}(L) \to \text{Set}(L)$ which is the functor for a monad whose $\eta$ captures the notion of singleton, and whose $\mu$ gives union. Similarly, the functor $T^{(1, 0)}_{(1, 0)}$ lifts $T_{1}$. A many-valued version of the ‘swapper’ (or distributive law) $\sigma$ allows us to construct the monad $\mathcal{L} \bullet T^{(1, 0)}_{(1, 0)}$ analogous to, but even more fully many-valued than $L_{id} \bullet T_{1}$.

### 5. An example

Let $\mathbb{N}_\text{set} = (\mathbb{N}, \Omega_{\mathbb{N}_\text{set}})$ be the signature of natural numbers (or the signature for the “algebra of natural numbers”), i.e. $\mathbb{N}_\text{set} = \{\text{naturals}\}$ and $\Omega_{\mathbb{N}_\text{set}} = \{0 : \to \text{naturals}, \text{succ : naturals} \to \text{naturals}\}$.

In $\text{Set}$, for $\Omega = \Omega_{\mathbb{N}_\text{set}}$ we have $\Omega_0 = \{0 : \to \text{naturals}\}$ and $\Omega_1 = \{\text{succ : naturals} \to \text{naturals}\}$. Further, $(\Omega_0)_{\text{set}} \times \text{id}^0A = \Omega_0$ and $(\Omega_1)_{\text{set}} \times \text{id}^1A = \{(1, \text{succ}, \alpha) \mid \alpha \in A\}$. For $T_{\Omega}$ we have $T^{0}_1A = A$ and $T^{1}_1A = \{\{(0, 0, (i)) \} \cup \{(1, \text{succ}, \alpha) \mid \alpha \in A\}$. From this we then continue to $T^{2}_1A = \{(0, 0, (i)) \cup \{(1, \text{succ}, \alpha) \mid \alpha \in A\}$, and $T^{3}_1A$ is unfolded similarly.

Correspondingly, in $\text{Set}(L)$, we have $(\Omega_0, \eta_{\text{set}}(L) \times \text{id}^0(A, \alpha) = (\Omega_0, \eta_0)$ and $(\Omega_1, \eta_1)_{\text{set}(L)} \times \text{id}^1(A, \alpha) = (\Omega_1, \eta_1) \times (A, \alpha) = (\Omega_1 \times A, \eta_1 \times \alpha)$, where $(\eta_1 \times \alpha)(\text{succ}, \alpha) = \eta_1(\text{succ}) \wedge \alpha(\alpha)$. For $T^{(1, 0)}_{(1, 0)}$ we then correspondingly have $T^{(1, 0)}_{(1, 0)}(A, \alpha) = (T^{0}_1A, \beta^0), \text{where } \beta^0 = \alpha$, and $T^{1}_{(1, 0)}(A, \alpha) = (\bigvee_{\kappa < 1} T^{\kappa}_{(1, 0)}(A, \alpha) = \bigvee_{\kappa < 2} T^{\kappa}_{(1, 0)}(A, \alpha))$.

### References


